MATH 110 – SOLUTIONS TO THE PRACTICE FINAL

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Note: There might be some mistakes and typos. Please let me know if you find any!

EXERCISE 1

Theorem: [Cauchy-Schwarz inequality]

Let V be a vector space over $\mathbb{F}$ with inner product $<,>$, and define $\|u\| = \sqrt{< u, u >}$.

Then for all u and v in V:

$$|< u, v >| \leq \|u\| \, \|v\|$$

Moreover, equality holds if and only if u is a multiple of v or v is a multiple of u

Proof:[1]

Fix u, v in V.

First of all, the result holds for $v = 0$, because:

$$< u, v > = < u, 0 > = 0 \leq \|u\| \, 0 = \|u\| \, \|v\|$$

And also if $v = 0$, then $v = 0u$, so v is a multiple of u

Hence, from now on, for the rest of the proof, we may assume $v \neq 0$.
Now consider $\boxed{< u - av, u - av >}$, where $a \in \mathbb{F}$ is to be selected later.

On the one hand, by the nonnegativity axiom of $<,>$, we have:

$$< u - av, u - av > \geq \; 0 \qquad (\star)$$

On the other hand, expanding $< u - av, u - av >$ out, we get:

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[1] The proof here is different from the one given in the book, but is equally valid. Feel free to memorize either one of them

$$\begin{aligned} < u - av, u - av > &= < u, u > + < u, -av > + < -av, u > + < -av, -av > \\ &= < u, u > -\overline{a} < u, v > -a < v, u > +a\overline{a} < v, v > \\ &= \|u\|^2 - \overline{a} < u, v > -a \overline{< u, v >} + |a|^2 \|v\|^2 \end{aligned}$$

Combining this with $(\star)$, we get:

$$\|u\|^2 + \overline{a} < u, v > +a < v, u > + |a|^2 \|v\|^2 \geq 0 \qquad (\star\star)$$

Now let $\boxed{a = \frac{<u,v>}{<v,v>} = \frac{<u,v>}{\|v\|^2}}$ [2] (which is well-defined since $v \neq 0$)

In particular, we get:

$$\overline{a} < u, v >= \frac{\overline{< u, v >} < u, v >}{\|v\|^2} = \frac{|< u, v >|^2}{\|v\|^2}$$

and

$$a\overline{< u, v >} = \frac{< u, v > \overline{< u, v >}}{\|v\|^2} = \frac{|< u, v >|^2}{\|v\|^2}$$

and

$$|a|^2 \|v\|^2 = \left(\frac{|< u, v >|}{\|v\|^2} \right)^2 \|v\|^2 = \frac{|< u, v >|^2}{\|v\|^2}$$

Therefore, $(\star\star)$ becomes:

$$\|u\|^2 - \frac{|< u, v >|^2}{\|v\|^2} - \cancel{\frac{|< u, v >|^2}{\|v\|^2}} + \cancel{\frac{|< u, v >|^2}{\|v\|^2}} \geq 0$$

That is:

$$\|u\|^2 - \frac{|< u, v >|^2}{\|v\|^2} \geq 0 \qquad (\star\star\star)$$

Solving for $< u, v >$, we get:

$$|< u, v >|^2 \leq \|u\|^2 \|v\|^2$$

And taking square roots (given that all the terms are nonnegative), we have:

$$|< u, v >| \leq \|u\| \|v\|$$

Which is the Cauchy-Schwarz inequality!

[2]The idea is that we want to turn $a < v, u >$ into $\frac{|<u,v>|^2}{\|v\|^2}$

Finally, if equality holds in the Cauchy-Schwarz inequality, that is, if $|< u, v >| = \|u\| \, \|v\|$ then working our way backwards, then $(\star\star\star)$ becomes an equality, and so $(\star\star)$ becomes an equality becomes equalities, and in particular, $(\star)$ becomes an equality, that is:

$$< u - av, u - av > = \ 0$$

And hence, by the positivity axiom, $u - av = 0$, that is, $u = av$, so u is a multiple of v.

Conversely, if $u = av$ for some $a \in \mathbb{F}$, then:

$$|< u, v >| = |< av, v >| = |a < v, v >| \, |a| \, |< v, v >| = |a| \, \|v\|^2 = |a| \, \|v\| \, \|v\| = \|av\| \, \|v\| = \|u\| \, \|v\|$$

So equality holds in the Cauchy-Schwarz inequality. Similarly if $v = au$ for some $a \in \mathbb{F}$ □

EXERCISE 2

Let $(v_1, \cdots, v_m)$ be a basis for $Nul(T)$, and extend it to a basis $(v_1, \cdots, v_m, v_{m+1}, \cdots, v_n)$ of V.

Claim: $(T(v_{m+1}), \cdots, T(v_n))$ is a basis of $Ran(T)$.

Proof: We need to show that the set spans $Ran(T)$ and is linearly independent

Span:

First of all, each $T(v_{m+1}), \cdots T(v_n)$ is in $Ran(T)$ (by definition of $Ran(T)$), and hence, because $Ran(T)$ is a subspace of W,

$$Span(T(v_{m+1}), \cdots, T(v_n)) \subseteq Ran(T)$$

Conversely, let $w \in Ran(T)$. Then $w = T(v)$ for some v in V.

But then, since $(v_1, \cdots, v_n)$ is a basis for V, we have $v = a_1 v_1 + \cdots + a_n v_n$ for scalars $a_1, \cdots, a_n$.

But then:

$$\begin{aligned} T(v) &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + \cdots + a_m T(v_m) + a_{m+1} T(v_{m+1}) + \cdots + a_n T(v_n) \\ &= a_1 0 + \cdots + a_m 0 + a_{m+1} T(v_{m+1}) + \cdots + a_n T(v_n) \quad \text{Because } v_1, \cdots, v_m \text{ are in } Nul(T) \\ &= a_{m+1} T(v_{m+1}) + \cdots a_n T(v_n) \\ &\in Span(T(v_{m+1}), \cdots, T(v_n)) \end{aligned}$$

Hence, $w = T(v) \in Span(T(v_{m+1}), \cdots, T(v_n))$.

Hence, since w was arbitrary, we get:

$$Ran(T) \subseteq Span(T(v_{m+1}), \cdots, T(v_n))$$

And therefore:

$$Span(T(v_{m+1}), \cdots, T(v_n)) = Ran(T)$$

So $T(v_{m+1}), \cdots, T(v_n)$ spans $Ran(T)$.

Linear independence:

Suppose $a_{m+1}T(v_{m+1}) + \cdots + a_nT(v_n) = 0$.

Then $T(a_{m+1}v_{m+1} + \cdots + a_nv_n) = 0$

Hence $a_{m+1}v_{m+1} + \cdots + a_nv_n \in Nul(T)$

Hence $a_{m+1}v_{m+1} + \cdots + a_nv_n = a_1v_1 + \cdots + a_mv_m$ for scalars $a_1, \cdots, a_m$, because $(v_1, \cdots, v_m)$ is a basis for $Nul(T)$.

Hence $-a_1v_1 - \cdots - a_mv_m + a_{m+1}v_{m+1} + \cdots + a_nv_n = 0$.

However, $(v_1, \cdots, v_n)$ is linearly independent, hence $-a_1 = \cdots = -a_m = a_{m+1} = \cdots = a_n = 0$
Hence $a_{m+1} = \cdots = a_n = 0$, which is what we wanted to show.

Hence $(T(v_{m+1}), \cdots, T(v_n))$ is a basis for $Ran(T)$, and hence $dim(Ran(T)) = n - m$

But then, it follows that:

$$dim(V) = n = m + (n - m) = dim(Nul(T)) + dim(Ran(T))$$

□

EXERCISE 3

Note: The explanations are optional, and are here to convince you why an answer is true or false.

(a) **VERY FALSE !!!** Remember that $Prop 1.9$ **ONLY** works for **TWO** subspaces!

(Let $V = \mathbb{R}^2$, $U_1 = Span\{(1,0)\}$ (the $x-$axis), $U_2 = Span\{(0,1)\}$ (the $y-$axis), and $U_3 = Span\{(1,1)\}$ (the line $y = x$). Notice that $V \neq U_1 \oplus U_2 \oplus U_3$ because $(0,0)$ can be written in two different ways as sums of vectors in U_1, U_2, U_3, namely $(0,0) = (1,0) + (0,-1) + (0,0)$, but also $(0,0) = (1,0) + (0,1) + (-1,-1)$. This violates the definition of direct sums on page 15)

(b) **FALSE**

(Let $V = \mathbb{R}^3$, and let $T \in \mathcal{L}(V)$ be the linear transformation whose matrix is $\mathcal{M}(T) = A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $(1,0)$ and $(0,1)$ are eigenvectors of T, but $(1,1) = (1,0) + (0,1)$ isn't)

(c) **TRUE**
(See Theorem 8.35. For a direct proof: Let m be the minimal polynomial of T and p be the characteristic polynomial. Then by the division algorithm for polynomials, there exist polynomials q and r with $deg(r) < deg(m)$ such that $m = qp + r$. But then $m(T) = q(T)p(T) + r(T)$. But $m(T) = 0$ by definition of the minimal polynomial, and $p(T) = 0$ by the Cayley-Hamilton theorem, whence $0 = 0 + r(T)$, so $r(T) = 0$, However, $deg(r) < deg(m)$, whence $r \equiv 0$ (otherwise this would contradict the definition of m as the *minimal* polynomial). But then $m = qp + r = qp + 0 = qp$, so p divides m)

(d) **TRUE**
(If $\mathbb{F} = \mathbb{R}$, then the real spectral theorem applies, and if $\mathbb{F} = \mathbb{C}$, then $T^*T = TT = TT^*$, so T is normal and the complex spectral theorem applies)

(e) **FALSE**
In general, the statement is false, and the reason is that we didn't specify whether the vector spacee is over $\mathbb{R}$ or over $\mathbb{C}$.

(In the case $\mathbb{F} = \mathbb{C}$, the answer is **FALSE**, because for example if $V = \mathbb{C}$, then $T(v) = iv$ has only one eigenvalue, $\lambda = i$, which is not real.

However, in the case $\mathbb{F} = \mathbb{R}$, the answer is **TRUE**, See Theorem 8.2 in section 8 of Axler's paper, or Theorem 5.26 on page 92 of the book.)

(f) **FALSE**

If you take the statement as it is, it doesn't make sense. It should be 'generalized eigenspaces of a (given) linear operator T'

Also, if you correct that statement, then it is **FALSE** if $\mathbb{F} = \mathbb{R}$, but **TRUE** if $\mathbb{F} = \mathbb{C}$.

(For the case $\mathbb{F} = \mathbb{C}$, this is just theorem 8.23 in the book. For the case $\mathbb{F} = \mathbb{R}$, consider $V = \mathbb{R}^2$, and let $T \in \mathcal{L}(V)$ be defined by $T(x, y) = (y, -x)$. Then $\mathcal{M}(T) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which has no (real) eigenvalues, and hence each eigenspace of T is just $\{0\}$. And hence the direct sum of the generalized eigenspaces of T is just $\{0\}$, which is not equal to $V = \mathbb{R}^2$)

EXERCISE 4

Since S is nilpotent, there exists m such that $S^m = 0$, and since T is nilpotent, there exists n such that $T^n = 0$.

Now, **because** $TS = ST$, the binomial formula applies, that is, for every k:

$$(S+T)^k = \sum_{j=0}^{k} a_j S^j T^{k-j}$$

(technically $a_j = \frac{j!k!}{(j-k)!}$, but we won't need that here)

Now take $\boxed{k = m+n}$, then:

$$\begin{aligned}(S+T)^{m+n} &= \sum_{j=0}^{m+n} a_j S^j T^{m+n-j} \\ &= \sum_{j=0}^{m} a_j S^j T^{m+n-j} + \sum_{j=m+1}^{n} S^j T^{m+n-j}\end{aligned}$$

However, if $j \leq m$, then $m+n-j \geq m+n-m = n$, so $m = n-j = n+l$, where $l \geq 0$, and so:

$$T^{m+n-j} = T^{n+l} = T^n T^l = 0T^l = 0$$

In particular $S^j T^{m+n-j} = S^j 0 = 0$, hence all the terms in the first sum are 0.

On the other hand, if $j \geq m+1$, then $j = m+l$, where $l \geq 0$, and so:

$$S^j = S^{m+l} = S^m S^l = 0S^l = 0$$

In particular, $S^j T^{m+n-j} = 0T^{m+n-j} = 0$, hence all the terms in the second sum are 0.

Combining this, we get:

$$(S+T)^{m+n} = 0$$

Hence $S+T$ is nilpotent.

EXERCISE 5

($\Leftarrow$) Suppose $x \in Nul(P)$ and $y \in Ran(P)$. We want to show that $< x, y >= 0$.

Since $x \in Nul(P)$, $P(x) = 0$, and since $y \in Ran(P)$, $y = P(z)$ for some $z \in V$.

But then:

$$< x, y > = < x, P(z) > = < P^* x, z > = < P(x), z >= < 0, z > = \ 0$$

Where in the third equality, we used the fact that $P^* = P$, since P is self-adjoint.

Hence $< x, y >= 0$, and we're done.

($\Rightarrow$) Suppose $Nul(P) \perp Ran(P)$. We want to show $P^* = P$. That is, for every x and y in V, we want to show that:

$$< Px, y > = < x, Py >$$

Notice that you can write:

$$< Px, y > = < Px, Py + (y - Py) >=< Px, Py > + < Px, y - Py >$$

However, notice that $Px \in Ran(P)$ and $P(y - Py) = Py - P^2 y = Py - Py = 0$, so $y - Py \in Nul(P)$.

Because $Nul(P) \perp Ran(P)$ by assumption, we get that $< Px, y - Py >= 0$. Therefore:

$$< Px, y >=< Px, Py >$$

But now, we can write:

$$< Px, y > = < Px, Py > = < x + (Px - x), Py > = < x, Py > + < Px - x, Py >$$

But $Py \in Ran(P)$ and $P(Px-x) = P^2 x - Px = Px - Py = 0$, so $Px - x \in Nul(P)$.

Because $Nul(P) \perp Ran(P)$ by assumption, we get that $< Px - x, Py >= 0$, and hence:

$$< Px, y > \ = \ < x, Py >$$

as we wanted to show □

EXERCISE 6

Note: Remember how to calculate the matrix of a linear transformation with respect to a basis $(v_1, \cdots, v_n)$! For each basis vector v_i, calculate $T(v_i)$ and express your result as a linear combination of *all* your basis vectors $(v_1, \cdots, v_n)$.

Here:

$$\begin{aligned} T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ &= \mathbf{1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{0}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \mathbf{3}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \mathbf{0}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Hence the first column of $\mathcal{M}(T)$ is:

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Next:

$$\begin{aligned} T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \\ &= \mathbf{0}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \mathbf{0}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \mathbf{3}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Hence the second column of $\mathcal{M}(T)$ is:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

Next:

$$\begin{aligned} T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} \\ &= \mathbf{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{0}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \mathbf{4}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \mathbf{0}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Hence the third column of $\mathcal{M}(T)$ is:

$$\begin{bmatrix} 2 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

Finally:

$$\begin{aligned} T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} \\ &= \mathbf{0} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \mathbf{0} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \mathbf{4} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Hence the fourth column of $\mathcal{M}(T)$ is:

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \end{bmatrix}$$

Putting everything together, we get:

$$\mathcal{M}(T) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

EXERCISE 7

($\Rightarrow$) Let $(v_1, \cdots, v_n)$ be a basis for V.

By definition, we know that $D(v_i) = \lambda_i v_i$ for some λ_i, where $i = 1, \cdots n$.

Now fix i and define $K_i \in \mathcal{L}(V)$ by:

$$\begin{aligned} K_i(v_1) &= 0 \\ &\vdots \\ K_i(v_{i-1}) &= 0 \\ K_i(v_i) &= v_i \\ K_i(v_{i+1}) &= 0 \\ &\vdots \\ K_i(v_n) &= 0 \end{aligned}$$

That is, $\boxed{K_i(v_j) = 0}$ for $j \neq i$, and $\boxed{K_i(v_i) = v_i}$.

Note that K_i exists by the linear extension lemma.

More explicitly (we'll need this below), if $v \in V$, then there exist $a_1, \cdots, a_n$ such that $v = a_1 v_1 + \cdots + a_n v_n$ (because $(v_1, \cdots, v_n)$ is a basis for V), and then:

$$\begin{aligned} K_i(v) &= K_i(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 K_i(v_1) + \cdots + a_i K_i(v_i) + \cdots + a_n K_i(v_n) \\ &= a_1 0 + \cdots + a_i v_i + \cdots + a_n 0 \\ &= a_i v_i \end{aligned}$$

That is, $\boxed{K_i(v) = a_i v_i}$, where $v = a_1 v_1 + \cdots + a_n v_n$

Now we only need to show 3 things:

(1) $D = \sum_{i=1}^{n} \lambda_i K_i$

Proof: Let $v \in V$, then $v = a_1 v_1 + \cdots + a_n v_n$ for scalars $i = 1, \cdots, n$.

But then:

$$
\begin{aligned}
D(v) &= D(a_1 v_1 + \cdots + a_n v_n) \\
&= a_1 D(v_1) + \cdots + a_n D(v_n) \\
&= a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n \\
&= \lambda_1 a_1 v_1 + \cdots + \lambda_n a_n v_n \\
&= \lambda_1 K_1(v) + \cdots + \lambda_n K_n(v) \\
&= \left(\sum_{i=1}^{n} \lambda_i K_i \right) v
\end{aligned}
$$

Since v was arbitrary, we get that $D = \sum_{i=1}^{n} \lambda_i K_i$

(2) $K_i^2 = K_i$

<u>Proof:</u> Let $v \in V$, then $v = a_1 v_1 + \cdots + a_n v_n$ for scalars $a_1, \cdots, a_n$.

But then:

$$
\begin{aligned}
K_i^2(v) &= K_i(K_i v) \\
&= K_i(K_i(a_1 v_1 + \cdots + a_n v_n)) \\
&= K_i(a_1 K_i(v_1) + \cdots + K_i(v_i) + \cdots + a_n K_i(v_n)) \\
&= K_i(a_1 0 + \cdots + a_i v_i + \cdots + a_n 0) \\
&= K_i(a_i v_i) \\
&= a_i K_i(v_i) \\
&= a_i v_i \\
&= K_i(v)
\end{aligned}
$$

Since v was arbitrary, we get $K_i^2 = K_i$ for all i

(3) If $i \neq j$, then $K_i K_j = 0$

<u>Proof:</u> Let $v \in V$, then $v = a_1 v_1 + \cdots + a_n v_n$ for scalars $a_1, \cdots, a_n$.

But then:

$$\begin{aligned} K_iK_j(v) &= K_i(K_jv) \\ &= K_i(K_j(a_1v_1 + \cdots + a_nv_n)) \\ &= K_i(a_1K_j(v_1) + \cdots + K_j(v_j) + \cdots + a_nK_j(v_n)) \\ &= K_i(a_1 0 + \cdots + a_jv_j + \cdots + a_n 0) \\ &= K_i(a_jv_j) \\ &= a_jK_i(v_j) \\ &= 0 \end{aligned}$$

(where in the last line we used $j \neq i$)

Hence $K_iK_j = 0$, since v was arbirary. □

($\Leftarrow$) For this, we use the result of exercise 11 in chapter 8 (which was on your homework), namely if $T \in \mathcal{L}(V)$, then:

$$V = Ran(T^n) \oplus Nul(T^n)$$

Here with $T = K_1$, we get:

$$V = Ran(K_1^n) \oplus Nul(K_1^n)$$

However, because $K_1^2 = K_1$, we have $K_1^n = K_1$ (use induction), and hence:

$$V = Ran(K_1) \oplus Nul(K_1)$$

Now let $U_1 = Nul(K_1)$. First of all, U_1 is invariant under K_2, because if $u_1 \in U_1$, then $K_1(K_2u_1) = (K_1K_2)u_1 = 0u_1 = 0$, so $K_2u_1 \in Nul(K_1) = U_1$. Hence, applying the result of exercise 11 in chapter 8 to $V = U_1$ and $T = K_2' := (K_2)\left|_{U_1}\right.$, we get:

$$Nul(K_1) = U_1 = Ran(K_2') \oplus Nul(K_2') \qquad (\star)$$

Claim: $Ran(K_2') = Ran(K_2)$

Proof: Suppose $v \in Ran(K_2)$, then $v = K_2(v')$ for some $v' \in V$. But then since $V = Ran(K_1) \oplus Nul(K_1)$, we get $v' = v_1 + v_2$, where $v_1 \in Ran(K_1)$, so $v_1 = K_1(u_1)$ and $v_2 \in Nul(K_1) = U_1$. But then

$$K_2(v') = K_2(v_1) + K_2(v_2) = K_2(K_1u_1) + K_2(v_2) = 0u_1 + K_2(v_2) = K_2(v_2) = K_2'v_2$$

That is: $v = K_2(v') = K_2'(v_2) \in Ran(K_2')$

Conversely, if $v \in Ran(K_2')$, then $v = K_2'(v')$ for some $v' \in U_1 \subseteq V$, so $v = K_2'(v') = K_2(v') \in Ran(K_2)$.

Claim: $Nul(K_2') = Nul(K_1) \cap Nul(K_2)$

Proof: Suppose $v \in Nul(K_1) \cap Nul(K_2)$, then $v \in Nul(K_1) = U_1$, and so $K_2'(v) = K_2(v) = 0$, since $v \in Nul(K_2)$, and so $v \in Nul(K_2')$.

Conversely, suppose $v \in Nul(K_2')$. Then $v \in U_1 = Nul(K_1)$ (by definition of K_2')), and hence $K_2 v = K_2' v = 0$, so $v \in Nul(K_2)$ as well, and hence $v \in Nul(K_1) \cap Nul(K_2)$

Combining the two claims and $(\star)$, we get:

$$U_1 = Ran(K_2) \oplus Nul(K_1) \cap Nul(K_2)$$

So if you let $U_2 = Nul(K_1) \cap Nul(K_2)$, you get:

$$U_1 = Ran(K_2) \oplus U_2$$

And so:

$$V = Ran(K_1) \oplus Ran(K_2) \oplus U_2$$

Now in general, you can prove by induction on i[3] that if $U_i = Nul(K_1) \cap \cdots \cap Nul(K_i)$, then:

$$U_i = Ran(K_i) \oplus U_{i+1}$$

And also by induction:

$$V = Ran(K_1) \oplus Ran(K_2) \oplus Ran(K_i) \oplus U_i$$

And in particular, with $i = n$, we get:

$$V = Ran(K_1) \oplus Ran(K_2) \oplus \cdots \oplus Ran(K_n) \oplus U_n$$

Where $U_n = Nul(K_1) \cap \cdots \cap Nul(K_n)$.
That is:

$$V = Ran(K_1) \oplus \cdots \oplus Ran(K_n) \oplus (Nul(K_1) \cap \cdots \cap Nul(K_n))$$

Note: If any of the above sets are 0, just delete them from the list.

Let $(v_1^i, \cdots, v_{k_i}^i)$ be a basis for $Ran(K_i)$ (where $k_i = dim(Ran(K_i))$) and $(w_1, \cdots, w_p)$ be a basis for $Nul(K_1) \cap \cdots \cap Nul(K_n)$.

Then because V is a direct sum of all the above spaces, we have that the whole list $(v_1^1, \cdots, v_{k_1}^1, \cdots, v_1^n, \cdots, v_{k_n}^n), w_1, \cdots, w_p)$ is a basis for V.

To show D is diagonal, we need (as usual) to calculate $D(v_j^k)$ and $D(w_i)$:

[3] do it!

First of all:

$$D(v_j^k) = (\sum_{i=1}^{m} \lambda_l K_i)(v_j^k) = \sum_{i=1}^{m} \lambda_i K_i(v_j^k)$$

Now $v_j^k \in Ran(K_k)$, so $v_j^k = K_k(u_j^k)$ for some $u_j^k)$, and hence:
$K_i(v_j^k) = K_i K_k(u_j^k)$, which is 0 if $i \neq k$, and if $i = k$, this is $K_k^2(u_j^k) = K_k(u_j^k) = v_j^k$

In other words, we get:

$$D(v_j^k) = \lambda_k v_j^k$$

Finally, for the w_i, notice that for all j, $K_j w_i = 0$ (because w_i is in the Nullspace of all the K_i), and so $D(w_i) = 0 = 0w_i$.

From this it follows that D is diagonal □